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# Analytical calculation of nonadiabatic transition probabilities from the monodromy of differential equations

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**Abstract**

The nonadiabatic transition probabilities in the two-level systems are calculated analytically by using the monodromy matrix determining the global feature of the underlying differential equation. We study the time-dependent $2 \times 2$ Hamiltonian with the tanh-type plus sech-type energy difference and with constant off-diagonal elements as an example to show the efficiency of the monodromy approach. We also discuss the application of this method to multi-level systems.

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## 1. Introduction

Since the beginning of quantum mechanics, the analytical calculation of the time evolution in two-level systems has been studied by a number of authors [1–14]. These results have been applied to various areas of physics including quantum optics, laser spectroscopy, nuclear magnetic resonance and atomic collisions [15–18]. The study of quantum time-evolution is still increasingly important even now. For example, much attention has been paid recently to the quantum manipulation of qubits [19] and the magnetization process of magnetic molecules with large spin [20]. Recent rapid development of computers has enabled massive numerical simulation of quantum dynamics. Nevertheless, it is still important to study solvable models analytically for the following reasons: (1) in some ranges of physical parameters the numerical simulation becomes too difficult, and (2) analytical solutions give a clearer description of parameter dependence.

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Analytical solutions of quantum dynamics can be classified into several classes, some of which are obtained by using hypergeometric functions. This was first discovered by Rosen and Zener [3], and has since been generalized by several other authors [7–12]. In these studies, the time variable $t$ is generally transformed into another real variable $z = z(t)$, which varies from 0 to 1 monotonically. Then, the Schrödinger equation of two-level systems can be reduced to the hypergeometric differential equation, and the transition probability can be related to the connection problem between two pairs of fundamental solutions around $z = 0$ and $z = 1$.

One exception is the approach by Carroll and Hioe [13], who studied two solvable classes; in one of these they have introduced a new variable $z(t)$ changing from $-\infty$ to $\infty$ as $t$ increases, and they have reduced the Schrödinger equation to the Riemann–Papperitz equation. Recently, Ishkhanyan [14] has pointed out that the Carroll–Hioe model can be understood in terms of the hypergeometric functions by considering a complex-valued path $z(t) = (y(t) + \mathrm{i})/2\mathrm{i}$ where $y(t)$ is a real variable. Using this complex-valued path, Ishkhanyan also found a new solvable class, but he did not obtain results for the transition probability.

In this paper, we show that for the complex-valued path, the transition probability can be calculated efficiently from the 'monodromy' matrices of the corresponding differential equations. Monodromy is one of the global properties of differential equations, and has attracted much attention from mathematicians, for example, through the deep connection with the Painlevé equations [21]. Hence, we expect that the monodromy approach is valuable not only because it enables us to calculate the transition probability for various models but also because it establishes a connection between physical phenomena and global features of differential equations.

In this paper, as a concrete example, we mainly consider the following time-dependent two-level Schrödinger equation and obtain the transition probability using the monodromy associated with the solution

$$\mathrm{i}\begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} = \begin{pmatrix} \varepsilon(t) & V(t) \\ V(t) & -\varepsilon(t) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1}$$

where the matrix elements are given by

$$\varepsilon(t) = E_0\,\mathrm{sech}(t/T) + E_1 \tanh(t/T) \tag{2}$$

$$V(t) = V_0. \tag{3}$$

Here, the coefficients, $E_0, E_1$ and $V_0$, are assumed to be real constants. This is one of the solvable classes reported by Ishkhanyan [14]. However, the transition probability for the model has not been obtained. It should be noted that this model is equivalent to the Rosen–Zener model [3] in the case $E_1 = 0$,[3] and that it also includes the special case of the second Demkov–Kunike model [7] in the case $E_0 = 0$. Hence, this model can give a smooth connection between the two known results.

The plan of the paper is as follows. We give the relation between the transition probability and the monodromy of the hypergeometric function in section 2, and the transition probability is calculated explicitly in section 3. The extension to the multi-level problems is addressed in section 4. Finally, the results are summarized in section 5. In appendix, we describe the generalization of the present model and its relationship to the Carroll–Hioe model.

[3] In the original Rosen–Zener model, $\varepsilon(t)$ is a constant, while $V(t)$ has a sech-type pulse form. This Hamiltonian, however, is reformed by a proper unitary transformation of the wavefunction to coincide with the present model.

## 2. Hypergeometric function and monodromy

The diagonal elements in the model (1) are eliminated by the following change of variables:

$$c_1 = a_1 \exp\left(\mathrm{i}\int_0^t \varepsilon\, \mathrm{d}t\right) \tag{4}$$

$$c_2 = a_2 \exp\left(-\mathrm{i}\int_0^t \varepsilon\, \mathrm{d}t\right). \tag{5}$$

Then, the Schrödinger equation is expressed as

$$\mathrm{i}c_{1t} = V \exp\left(2\mathrm{i}\int_0^t \varepsilon\, \mathrm{d}t\right) c_2 \tag{6}$$

$$\mathrm{i}c_{2t} = V \exp\left(-2\mathrm{i}\int_0^t \varepsilon\, \mathrm{d}t\right) c_1. \tag{7}$$

By combining these two equations, we obtain the second-order differential equations for $c_1$ and $c_2$, respectively, as

$$c_{1tt} + \left(-2\mathrm{i}\varepsilon(t) - \frac{V_t}{V}\right) c_{1t} + V^2 c_1 = 0 \tag{8}$$

$$c_{2tt} + \left(2\mathrm{i}\varepsilon(t) - \frac{V_t}{V}\right) c_{2t} + V^2 c_2 = 0. \tag{9}$$

It should be noted that the equation for $c_2$ is obtained by replacing $\varepsilon(t)$ by $-\varepsilon(t)$ in equation (8). Hence, once the solution of the equation for $c_1$ is obtained, the solution for $c_2$ is easily obtained by reversing the sign of the parameters in $\varepsilon(t)$.

The above discussion is general. Now, we consider the specific model given by equations (2) and (3). By substituting these specific forms of $\varepsilon(t)$ and $V(t)$ into equation (8) and adopting the change of variable as

$$z(t) = \frac{\sinh(t/T) + \mathrm{i}}{2\mathrm{i}} \tag{10}$$

equation (8) can be reduced to the differential equation of the hypergeometric function [22]

$$z(1-z)c_{1zz} + (\gamma - (1+\alpha+\beta)z)c_{1z} - \alpha\beta c_1 = 0. \tag{11}$$

Here, the parameters, $\alpha$, $\beta$ and $\gamma$, are determined as

$$\alpha = \mathrm{i}T\left(-E_1 + \sqrt{E_1^2 + V_0^2}\right) \tag{12}$$

$$\beta = \mathrm{i}T\left(-E_1 - \sqrt{E_1^2 + V_0^2}\right) \tag{13}$$

$$\gamma = \tfrac{1}{2} + E_0 T - \mathrm{i}E_1 T. \tag{14}$$

In the same way, equation (9) is reduced to the hypergeometric differential equation with the parameters

$$\alpha' = \mathrm{i}T\left(E_1 + \sqrt{E_1^2 + V_0^2}\right) \tag{15}$$

$$\beta' = \mathrm{i}T\left(E_1 - \sqrt{E_1^2 + V_0^2}\right) \tag{16}$$

$$\gamma' = \tfrac{1}{2} - E_0 T + \mathrm{i}E_1 T. \tag{17}$$

As already mentioned, these parameters are obtained by replacing $E_0$ and $E_1$ by $-E_0$ and $-E_1$ respectively in equations (12)–(14). In the following calculation, related to the calculation for $c_2$, the prime indicates that it is obtained by reversing the sign of $E_0$ and $E_1$ from the original quantity without the prime.

From equation (10), it can be easily seen that the variable $|z| \to \infty$ as $|t| \to \infty$. Hence, for the discussion about the initial state it is convenient to consider the fundamental solutions around $z = \infty$ as

$$c_1 = A_1 f_\infty(z;\alpha) + A_2 f_\infty(z;\beta) \tag{18}$$

$$c_2 = B_1 f_\infty(z;\alpha') + B_2 f_\infty(z;\beta') \tag{19}$$

where $f_\infty(z;\alpha)$ and $f_\infty(z;\beta)$ are expressed in terms of the hypergeometric functions as

$$f_\infty(z;\alpha) = z^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; 1/z) \tag{20}$$

$$f_\infty(z;\beta) = z^{-\beta} F(\beta - \gamma + 1, \beta, \beta - \alpha + 1; 1/z) \tag{21}$$

with similar definitions for $f_\infty(z;\alpha')$ and $f_\infty(z,\beta')$. Since $\alpha$ and $\beta$ are pure-imaginary in the present model, we have to choose $\arg(z)$ to determine the branch. In this paper, we choose

$$\arg(z) = \begin{cases} \pi/2 & (t \to -\infty) \\ -\pi/2 & (t \to +\infty). \end{cases} \tag{22}$$

In order to decide the initial state, it is sufficient to study the limit $|z| \to \infty$, in which case we obtain

$$c_1 \to A_1 z^{-\alpha} + A_2 z^{-\beta} \tag{23}$$

$$c_2 \to B_1 z^{-\alpha'} + B_2 z^{-\beta'}. \tag{24}$$

From equations (10), (23) and (24), $A_i$ and $B_i$ are determined.

To obtain the transition probability, we assume that the initial state is the ground state of the Hamiltonian in the limit $t \to -\infty$

$$H = \begin{pmatrix} -E_1 & V_0 \\ V_0 & E_1 \end{pmatrix}. \tag{25}$$

It may be noted that the off-diagonal elements in equation (25) do not vanish. Consequently, the ground-state wavefunction does not correspond to $|a_1| = 1$ and $a_2 = 0$ as it appears in the usual models. The time evolution of the ground-state wavefunction is obtained generally as

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} A \\ -A' \end{pmatrix} \mathrm{e}^{-\mathrm{i}(-\sqrt{E_1^2+V_0^2})t+\mathrm{i}\varphi} \tag{26}$$

where

$$A = \sqrt{\frac{E_1 + \sqrt{E_1^2 + V_0^2}}{2\sqrt{E_1^2 + V_0^2}}} \qquad A' = \sqrt{\frac{-E_1 + \sqrt{E_1^2 + V_0^2}}{2\sqrt{E_1^2 + V_0^2}}}. \tag{27}$$

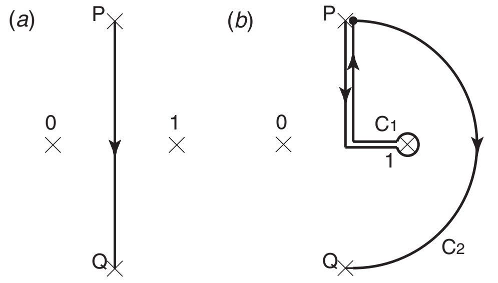


**Figure 1.** The path along which the analytical continuation is performed: (*a*) the original path; (*b*) the deformed path.

The solution (26) includes an arbitrary phase factor $\varphi$, which is chosen to be zero in this paper. From equation (26), the time evolutions of $c_1$ and $c_2$ in the limit $t \to -\infty$ can be easily evaluated as

$$c_1(t) = a_1(t) \exp\left(\mathrm{i} \int_0^t \varepsilon \, \mathrm{d}t\right) \to A \, \mathrm{e}^{-\mathrm{i}(E_1 - \sqrt{E_1^2 + V_0^2})t - \mathrm{i}\phi_0 - \mathrm{i}\phi_1} \tag{28}$$

$$c_2(t) = a_2(t) \exp\left(-\mathrm{i} \int_0^t \varepsilon \, \mathrm{d}t\right) \to -A' \, \mathrm{e}^{-\mathrm{i}(-E_1 - \sqrt{E_1^2 + V_0^2})t + \mathrm{i}\phi_0 + \mathrm{i}\phi_1} \tag{29}$$

where the phase factors are given as

$$\phi_0 = T E_0 \frac{\pi}{2} \qquad \phi_1 = T E_1 \log 2. \tag{30}$$

By using equation (10) and by comparing equations (28)–(29) with equations (23)–(24), the constants $A_i$ and $B_i$ are determined as

$$A_1 = A \, \mathrm{e}^{\mathrm{i}\pi\alpha/2 - \mathrm{i}\phi_1 - \mathrm{i}\phi_0 - \mathrm{i}\varphi_1} \qquad A_2 = 0 \tag{31}$$

$$B_1 = -A' \, \mathrm{e}^{\mathrm{i}\pi\alpha'/2 + \mathrm{i}\phi_1 + \mathrm{i}\phi_0 - \mathrm{i}\varphi_1'} \qquad B_2 = 0 \tag{32}$$

where the phase factors, $\varphi_1$ and $\varphi_1'$, are given as

$$\mathrm{i}\varphi_1 = 2\alpha \log 2 \qquad \mathrm{i}\varphi_1' = 2\alpha' \log 2 \tag{33}$$

although these are not relevant to the calculation of the transition probability.

By the choice of the initial condition, the time evolution has been described only by the fundamental solution $f_\infty(z; \alpha)$ around $z = \infty$. To be more accurate, around $z = \mathrm{i}\infty + 1/2$ (corresponding to $t \to -\infty$) denoted by the point P in figure 1(*a*), the solution is given by

$$c_1(z) = A_1 f_\infty(z; \alpha) \tag{34}$$

$$c_2(z) = B_1 f_\infty(z; \alpha'). \tag{35}$$

On the other hand, the final state is given by the solution of the hypergeometric differential equation around $z = -\mathrm{i}\infty + 1/2$ (corresponding to $t \to \infty$) denoted by the point Q in figure 1(*a*). The path $z(t)$ in the complex plane is also drawn in figure 1. Then, the solution around the point Q analytically continued from the point P does not equal equations (34) and (35); the solution is expressed as linear combinations of the fundamental solutions around $z = \infty$. This is crucial to the calculation of the transition probability.

In order to make the situation more clear, let us deform the path of $z$ as shown in figure 1(*b*). In this deformed path, the analytical continuation of the solution is divided into two parts, $C_1$ and $C_2$. Here, the path $C_1$ denotes a round trip to the singular point at $z = 1$, while the path $C_2$ is a half round trip around $z = \infty$ in the clockwise direction. The analytical continuation along the path $C_2$ is easily performed, and determined only by the fundamental solutions around $z = \infty$. On the other hand, the analytical continuation along the path $C_1$ is nontrivial, and determined by the global character of the differential equation called the 'monodromy'. The monodromy is expressed by the monodromy matrices as

$$\gamma(C_1)(f_\infty(z;\alpha), f_\infty(z;\beta)) = (f_\infty(z;\alpha), f_\infty(z;\beta))R \tag{36}$$

$$\gamma(C_1)(f_\infty(z;\alpha'), f_\infty(z;\beta')) = (f_\infty(z;\alpha'), f_\infty(z;\beta'))R' \tag{37}$$

where $\gamma(C_1)$ denotes the analytical continuation along the path $C_1$. Denoting the matrix elements of $R$ and $R'$ as

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad R' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{38}$$

the solutions around the point Q can be expressed by

$$\begin{aligned}\gamma(C_1)(c_1(z)) &= \gamma(C_1)(A_1 f_\infty(z;\alpha)) \\ &= A_1 a f_\infty(z;\alpha) + A_1 c f_\infty(z;\beta)\end{aligned} \tag{39}$$

$$\begin{aligned}\gamma(C_1)(c_2(z)) &= \gamma(C_1)(B_1 f_\infty(z;\alpha')) \\ &= B_1 a' f_\infty(z;\alpha') + B_1 c' f_\infty(z;\beta').\end{aligned} \tag{40}$$

Here, as shown below, the first (second) term in the final entries of equations (39) and (40) corresponds to the excited (ground) state in the limit $t \to \infty$. Hence, in the calculation of the transition probability, only the element $a$ ($a'$) is relevant. This matrix element is calculated explicitly in the next section.

Let us end this section by deriving the formula for the transition probability, using the monodromy matrix elements. In the limit $t \to \infty$, equations (39) and (40) are evaluated as

$$c_1(t) \to aA_1\,\mathrm{e}^{\mathrm{i}\varphi+\mathrm{i}\pi\alpha/2}\,\mathrm{e}^{-\mathrm{i}(E_1+\sqrt{E_1^2+V_0^2})t} + \cdots \tag{41}$$

$$c_2(t) \to aB_1\,\mathrm{e}^{\mathrm{i}\varphi'+\mathrm{i}\pi\alpha'/2}\,\mathrm{e}^{-\mathrm{i}(-E_1+\sqrt{E_1^2+V_0^2})t} + \cdots. \tag{42}$$

Here, we have suppressed the second term corresponding to the ground state. From these equations, the components of the wavefunction $\Psi(t) = (a_1(t), a_2(t))^{\mathrm{T}}$ is obtained in the limit $t \to \infty$ as

$$\begin{aligned}a_1(t) &= c_1(t)\exp\left\{-\mathrm{i}\int_0^t \varepsilon(t)\,\mathrm{d}t\right\} \\ &\to aA_1\,\mathrm{e}^{\mathrm{i}\varphi+\mathrm{i}\pi\alpha+\mathrm{i}\phi_1-\mathrm{i}\phi_0}\,\mathrm{e}^{-\mathrm{i}\sqrt{E_1^2+V_0^2}t} + \cdots\end{aligned} \tag{43}$$

$$\begin{aligned}a_2(t) &= c_2(t)\exp\left\{+\mathrm{i}\int_0^t \varepsilon(t)\,\mathrm{d}t\right\} \\ &\to a'B_1\,\mathrm{e}^{\mathrm{i}\varphi'+\mathrm{i}\pi\alpha'-\mathrm{i}\phi_1+\mathrm{i}\phi_0}\,\mathrm{e}^{-\mathrm{i}\sqrt{E_1^2+V_0^2}t} + \cdots.\end{aligned} \tag{44}$$

By substituting equations (31) and (32), the wavefunction $\Psi(t)$ is evaluated as

$$\Psi(t) \to \begin{pmatrix} aA\,\mathrm{e}^{-2\mathrm{i}\phi_0+\mathrm{i}\pi\alpha} \\ -a'A'\,\mathrm{e}^{2\mathrm{i}\phi_0+\mathrm{i}\pi\alpha'} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\sqrt{E_1^2+V_0^2}t} + \cdots. \tag{45}$$

On the other hand, the Hamiltonian and the wavefunction of the excited state $\Psi_{\mathrm{E.S.}}$ in the limit $t \to \infty$ are given as

$$H = \begin{pmatrix} E_1 & V_0 \\ V_0 & -E_1 \end{pmatrix} \qquad \Psi_{\mathrm{E.S.}} = \begin{pmatrix} A \\ A' \end{pmatrix} \tag{46}$$

where $A$ and $A'$ are given by equation (27). Then, the transition probability is calculated as

$$\begin{aligned} P &= |\Psi(t \to \infty) \cdot \Psi_{\mathrm{E.S.}}|^2 \\ &= |aA^2\,\mathrm{e}^{\mathrm{i}\pi\alpha - 2\mathrm{i}\phi_0} - a'A'^2\,\mathrm{e}^{\mathrm{i}\pi\alpha' + 2\mathrm{i}\phi_0}|^2 \\ &= \Bigg| \frac{\varepsilon_1}{2\sqrt{\varepsilon_1^2 + v^2}} \big(a\,\mathrm{e}^{\varepsilon_1 - \sqrt{\varepsilon_1^2+v^2}}\,\mathrm{e}^{-\mathrm{i}\varepsilon_0} + a'\,\mathrm{e}^{-\varepsilon_1 - \sqrt{\varepsilon_1^2+v^2}}\,\mathrm{e}^{\mathrm{i}\varepsilon_0}\big) \\ &\quad + \frac{1}{2}\big(a\,\mathrm{e}^{\varepsilon_1 - \sqrt{\varepsilon_1^2+v^2}}\,\mathrm{e}^{-\mathrm{i}\varepsilon_0} - a'\,\mathrm{e}^{-\varepsilon_1 - \sqrt{\varepsilon_1^2+v^2}}\,\mathrm{e}^{\mathrm{i}\varepsilon_0}\big)\Bigg|^2 \end{aligned} \tag{47}$$

where in the final equation we have introduced new variables

$$\varepsilon_0 = \pi T E_0 \qquad \varepsilon_1 = \pi T E_1 \qquad v = \pi T V_0. \tag{48}$$

Equation (47) can be used for the practical evaluation of the transition probability. The remaining task is to calculate the elements of the monodromy matrices.

## 3. Calculation of the transition probability

We may identify several ways to calculate the monodromy matrices of the hypergeometric differential equations [21]. Here, we briefly explain the simplest way.

To determine the monodromy matrix, it is crucial to use the integral representation of the hypergeometric function. By defining the integral

$$F_{pq}(z) = \int_p^q \mathrm{d}t\, t^{\alpha-\gamma}(1-t)^{\gamma-\beta-1}(z-t)^{-\alpha} \tag{49}$$

the following relations hold

$$F_{1\infty} = c_{1\infty} f_0(z; 0) \qquad F_{0z} = c_{0z} f_0(z; 1-\gamma) \tag{50}$$

$$F_{\infty 0} = c_{\infty 0} f_1(z; 0) \qquad F_{1z} = c_{1z} f_1(z; \gamma - \alpha - \beta) \tag{51}$$

$$F_{01} = c_{01} f_\infty(z; \alpha) \qquad F_{z\infty} = c_{z\infty} f_\infty(z; \beta) \tag{52}$$

where $f_0$, $f_1$ and $f_\infty$ denote the fundamental solutions of the hypergeometric differential equations around $z = 0, 1, \infty$, respectively. Here, the constants, $c_{pq}$, depend only on $\alpha$, $\beta$ and $\gamma$, and their explicit expressions are irrelevant to the present calculation. By applying the Cauchy theorem to the integral in equation (49), the following linear relations may be identified

$$F_{01} + F_{1\infty} + F_{\infty 0} = 0 \tag{53}$$

$$F_{01} - F_{0z} + F_{1z} = 0 \tag{54}$$

$$\mathrm{e}(\beta - \gamma + 1)F_{1\infty} - F_{1z} - \mathrm{e}(-\alpha)F_{z\infty} = 0 \tag{55}$$

$$\mathrm{e}(\alpha - \gamma)F_{\infty 0} + F_{0z} + F_{z\infty} = 0 \tag{56}$$

where $\mathrm{e}(\cdot) = \exp(2\pi\mathrm{i}\cdot)$. By eliminating $F_{1\infty}$ and $F_{0z}$, we obtain

$$(F_{01}, F_{z\infty}) = (F_{\infty 0}, F_{1z})S \tag{57}$$

$$S = \frac{1}{\mathrm{e}(-\alpha) - \mathrm{e}(\beta-\gamma)} \begin{pmatrix} \mathrm{e}(\beta-\gamma) - \mathrm{e}(-\gamma) & \mathrm{e}(\alpha+\beta-2\gamma) - \mathrm{e}(\beta-\gamma) \\ 1 - \mathrm{e}(-\alpha) & \mathrm{e}(\beta-\gamma) - 1 \end{pmatrix}. \tag{58}$$

On the other hand, since the solution pair $(F_{\infty 0}, F_{1z})$ is related to the fundamental solutions around $z = 1$, the monodromy matrix for this pair is easily obtained as

$$\gamma(\mathrm{C}_1)(F_{\infty 0}, F_{1z}) = (F_{\infty 0}, F_{1z})\Gamma \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}(\gamma-\alpha-\beta) \end{pmatrix}. \tag{59}$$

Combining equations (58) and (59), the monodromy matrix for $(F_{\infty 0}, F_{1z})$ is given as

$$\gamma(\mathrm{C}_1)(F_{01}, F_{z\infty}) = (F_{01}, F_{z\infty})\tilde{R} \tag{60}$$

$$\tilde{R} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = S^{-1}\Gamma S. \tag{61}$$

This result is easily related to the fundamental solutions, $f_\infty(z;\alpha)$ and $f_\infty(z;\beta)$, by equation (52) as

$$\gamma(\mathrm{C}_1)\left(f_\infty(z,\alpha), \frac{c_{z\infty}}{c_{01}} f_\infty(z,\beta)\right) = \left(f_\infty(z,\alpha), \frac{c_{z\infty}}{c_{01}} f_\infty(z,\beta)\right)\tilde{R}. \tag{62}$$

By comparing the above with the original monodromy matrix (36) along with equation (38), we finally obtain $a = \tilde{a}$ $(d = \tilde{d})$. So, as far as $a$ is concerned, we only need to calculate $\tilde{R}$. This can be performed by a straightforward but slightly lengthy calculation. As a result, we obtain the matrix element $a$ as

$$a = \frac{\mathrm{e}(\beta-\gamma) - \mathrm{e}(-\gamma) + \mathrm{e}(-\alpha) - 1}{\mathrm{e}(\beta-\gamma) - \mathrm{e}(\alpha-\gamma)}. \tag{63}$$

The matrix element $a'$ for the solution $c_2$ is easily obtained by reversing the sign of $E_0$ and $E_1$ in the result (63). From the result for $a$ and $a'$, the transition probability $P$ is obtained from equation (47) as

$$P = \frac{\sinh^2(\pi T E_1)\cos^2(\pi T E_0)}{\sinh^2\left(\pi T\sqrt{E_1^2 + V_0^2}\right)} + \frac{\cosh^2(\pi T E_1)\sin^2(\pi T E_0)}{\cosh^2\left(\pi T\sqrt{E_1^2 + V_0^2}\right)}. \tag{64}$$

Let us discuss the nature of this result. The transition probability oscillates as the sech-form pulse area, $\pi T E_0$, changes. The transition probability has minimum and maximum values as a function of $E_0$ as

$$\begin{aligned} P_{\min} &= \frac{\sinh^2(\pi T E_1)}{\sinh^2\left(\pi T\sqrt{E_1^2 + V_0^2}\right)} \qquad \text{for} \quad \pi T E_0 = n\pi \\ P_{\max} &= \frac{\cosh^2(\pi T E_1)}{\cosh^2\left(\pi T\sqrt{E_1^2 + V_0^2}\right)} \qquad \text{for} \quad \pi T E_0 = (n+1/2)\pi \end{aligned} \tag{65}$$

where $n$ is an integer. The oscillation behaviour of $P$ for $E_0$ is shown in figure 2(*a*).

The amplitude of this oscillation becomes small as $E_1$ increases. This feature is shown in figure 2(*b*). For $E_1 T \gg \max(V_0 T, 1)$, we obtain the ordinary Landau–Zener formula

$$P = \mathrm{e}^{-\pi V_0^2 T / 2E_1} \tag{66}$$

independent of $E_0$.

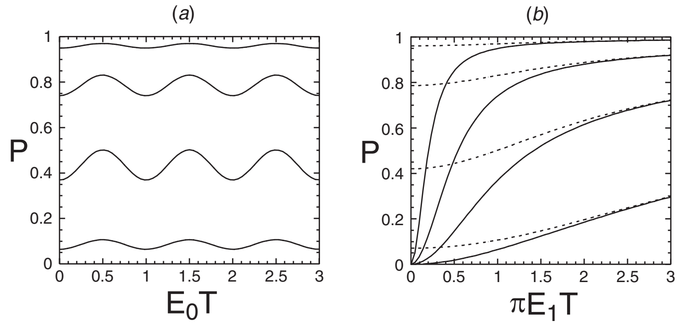


**Figure 2.** (*a*) The $E_0$ dependence of the transition probability $P$ for $\pi T E_1 = 1$. From top to bottom, $\pi V_0 T = 0.2$, 0.5, 1.0 and 2.0. (*b*) The $E_1$ dependence of $P$. From top to bottom, $\pi V_0 T = 0.2$, 0.5, 1.0 and 2.0. The solid (dashed) lines show the minimum (maximum) value of $P$ at each $E_1$.

Finally, we show that the results in the limiting cases coincide with the known results. In the limit $E_1 \to 0$, the transition probability is given as

$$P = \frac{\sin^2(\pi T E_0)}{\cosh^2(\pi T V_0)} \tag{67}$$

which corresponds to the Rosen–Zener formula [3]. In the limit $E_0 \to 0$, the transition probability is given as

$$P = \frac{\sinh^2(\pi T E_1)}{\sinh^2\left(\pi T\sqrt{E_1^2 + V_0^2}\right)}. \tag{68}$$

In this case, the present model is related to the second model in the paper by Demkov and Kunike [7], which corresponds to the form

$$\varepsilon(t) = a + b\tanh(t/T) \tag{69}$$

$$V(t) = c. \tag{70}$$

Their result for $a = 0$ corresponds to the result (68).

## 4. Application of monodromy to multi-level problems

The application of the monodromy matrix to the transition probability is not restricted to hypergeometric functions. The monodromy approach is also applicable to differential equations whose monodromy is known. To show such an example, we consider the multi-level problem. We expect that more solvable classes can be found by using the present approach.

In this section, we treat the following time-dependent Hamiltonian

$$H_{ij} = \begin{cases} \varepsilon(t) & (i = j = 1) \\ V_j & (i = 1 \text{ and } 2 \leqslant j \leqslant N) \\ V_i & (j = 1 \text{ and } 2 \leqslant i \leqslant N) \\ 0 & (\text{otherwise}) \end{cases} \tag{71}$$

where the time-dependent part $\varepsilon(t)$ is given as

$$\varepsilon(t) = E_1 \tanh(t/T) \tag{72}$$

and $V_j$ ($2 \leqslant j \leqslant N$) are constants. It should be noted that in the limit $E_1 T \to \infty$, this model is reduced to the extended Landau–Zener model studied by several authors [23–26]. To eliminate the diagonal element of the Hamiltonian the wavefunction denoted by $\Psi(t) = (a_1, a_2, \ldots, a_N)^{\mathrm{T}}$ is transformed into new variables as

$$c_i = \begin{cases} a_1 \exp\left(\mathrm{i}\int_0^t \varepsilon\,\mathrm{d}t\right) & (i = 1) \\ a_i & (2 \leqslant i \leqslant N). \end{cases} \tag{73}$$

The integral in the exponent is then calculated as

$$\mathrm{i}\int_0^t \varepsilon\,\mathrm{d}t = \mathrm{i}E_1 T \log(\cosh t/T). \tag{74}$$

Thus, the Schrödinger equation is obtained as

$$T c_{i,t} = \begin{cases} \sum_{j=2}^{N} v_j (\cosh t/T)^{2\varepsilon_1} c_j & (i = 1) \\ v_i (\cosh t/T)^{-2\varepsilon_1} c_1 & (2 \leqslant i \leqslant N) \end{cases} \tag{75}$$

where

$$\varepsilon_1 = \mathrm{i}E_1 T/2 \qquad v_j = -\mathrm{i}V_j T. \tag{76}$$

By changing the time variable as $z = \sinh(t/T)$, the equations are modified as

$$\frac{\mathrm{d}c_i}{\mathrm{d}z} = \begin{cases} \sum_{j=2}^{N} v_j (1 + z^2)^{\varepsilon_1 - 1/2} c_j & (i = 1) \\ v_i (1 + z^2)^{-\varepsilon_1 - 1/2} c_1 & (2 \leqslant i \leqslant N) \end{cases}. \tag{77}$$

We make a further change of variables as

$$d_i = \begin{cases} (1 + z^2)^{-\varepsilon_1 - 1/2} c_1 & (i = 1) \\ \frac{v_i c_i}{z + \mathrm{i}} - \lambda_i \left(\varepsilon_1 + \frac{1}{2}\right) \frac{z - \mathrm{i}}{z + \mathrm{i}} d_1 & (2 \leqslant i \leqslant N) \end{cases} \tag{78}$$

where $\lambda_j$ are arbitrary constants satisfying

$$\sum_{j=2}^{N} \lambda_j = 1. \tag{79}$$

Consequently, we finally obtain

$$(z - \mathrm{i})\frac{\mathrm{d}d_1}{\mathrm{d}z} = -\left(\varepsilon_1 + \frac{1}{2}\right) d_1 + \sum_{j=2}^{N} d_j \tag{80}$$

and for $2 \leqslant i \leqslant N$

$$(z + \mathrm{i})\frac{\mathrm{d}d_i}{\mathrm{d}z} = \lambda_i \left(\varepsilon_1^2 + v_i^2 - \frac{1}{4}\right) d_1 - d_i - \lambda_i \left(\varepsilon_1 + \frac{1}{2}\right) \sum_{j=2}^{N} d_j. \tag{81}$$

This is the Okubo equation expressed by

$$(zI - C)\frac{\mathrm{d}\vec{d}}{\mathrm{d}z} = A\vec{d} \tag{82}$$

where $I$ is the identity operator, $C$ is a diagonal matrix, and $A$ is a general matrix. This equation has been studied in detail by Okubo [27], and it is known that this form of equation is convenient to study the monodromy.

Thus, it has been shown that at least one specific model of multi-level systems can be reduced to the differential equation whose monodromy is known. The actual calculation of the transition probability needs explicit treatment of the monodromy matrices, and remains as a future problem. The present discussion for multi-level systems is preliminary, and more detailed study will be needed to clarify the efficiency of the monodromy approach.

## 5. Summary

We have calculated the transition probability for the Hamiltonian including the tanh-type plus sech-type energy difference with constant off-diagonal elements. The obtained result gives the natural connection between the known results, the Rosen–Zener model and the second Demkov–Kunike model. This model also includes the Landau–Zener formula in the limit of the large amplitude of the tanh-type energy difference.

In our calculation, the monodromy of the hypergeometric functions is essential. We have shown that the monodromy approach is also applicable to the multi-level problems. We expect that the use of the monodromy in the calculation of the transition probability not only helps in finding more solvable models but also connects global properties of the differential equation with the physical phenomena. Details of the calculation especially for the multi-level problem remain problems for future work.

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## Appendix. Solvable classes

The model considered in the main part of this paper belongs to one solvable class called class 1 below. It can be given as

$$\varepsilon(t) = \frac{E_0 T + E_1 T y}{1 + y^2} \frac{\mathrm{d}y}{\mathrm{d}t} \tag{A.1}$$

$$V(t) = \frac{V_0 T}{\sqrt{1 + y^2}} \frac{\mathrm{d}y}{\mathrm{d}t} \tag{A.2}$$

where $y(t)$ is an ‘arbitrary’ monotonically increasing function satisfying $y(t) \to \pm\infty$ for $t \to \pm\infty$. When we adopt $y(t) = \sinh(t/T)$, we obtain equations (2) and (3). For this class, the Schrödinger equation can be reduced to the same hypergeometric differential equation (11) through the change of variable $z(t) = (y(t) + \mathrm{i})/2\mathrm{i}$ [14]. Hence, all models of this class give the same transition probability (64). In this class, however, we have to define the transition probability carefully. In the limit $t \to -\infty$ ($y \to -\infty$), the matrix elements become

$$\varepsilon(t) \to \frac{E_1 T}{y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{A.3}$$

$$V(t) \to -\frac{V_0 T}{y}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{A.4}$$

Hence, the wavefunction of the ground state in this limit has mixed components as treated in section 2. The initial state is taken as the ground state in this limiting Hamiltonian, and the transition probability is defined as the square of modulus of the final amplitude of the excited states.

The application of the monodromy is not restricted to class 1. As discussed by Ishkhanyan [14], as long as the complex path $z(t) = (y(t)+\mathrm{i})/2\mathrm{i}$ is used, the calculation by the monodromy is efficient. For example, the following solvable class can be considered:

$$\varepsilon(t) = \frac{E_0 T + E_1 T y}{1 + y^2}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{A.5}$$

$$V(t) = \frac{V_0 T}{1 + y^2}\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{A.6}$$

This class, here called class 2, was first studied by Carroll and Hioe [13]. There, the transition probability was calculated by solving the Riemann–Papperitz equation without resorting to the monodromy. By following the discussion of Ishkhanyan [14], however, our monodromy approach is also efficient for class 2, and gives an alternative method.

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